Differential Geometry Chapter 1

Vectors

$\mathbb{R}^n$ is Euclidean n-space, the set of all ordered n-tuples of real numbers. We will write vectors vertically.

Definition 1 *A **tangent vector** $\mathbf{v_p}$ to $\mathbb{R}^n$ consists of two parts, a vector $\mathbf{v} = (v_1, ..., v_n)^T \in \mathbb{R}^n$ and a point of application $\mathbf{p} \in \mathbb{R}^n$.*

So $\mathbf{v_p} = \mathbf{w_q}$ iff $\mathbf{v} = \mathbf{w}$ and $\mathbf{p} = \mathbf{q}$. If $\mathbf{v} = \mathbf{w}$ but $\mathbf{p} \neq \mathbf{q}$ then $\mathbf{v_p}$ and $\mathbf{w_p}$ are **parallel**.

Definition 2 *The collection of all vectors having their point of application at $\mathbf{p}$ is the **Tangent Space of** $\mathbb{R}^n$ **at** $\mathbf{p}$ denoted by $T_{\mathbf{p}}(\mathbb{R}^n)$.*

Make $T_{\mathbf{p}}(\mathbb{R}^n)$ into a vector space by

Definition 3 $\mathbf{v_p} + \mathbf{w_p} = (\mathbf{v} + \mathbf{w})_p$ *and* $\lambda \mathbf{v_p} = (\lambda \mathbf{v})_p$ *for* $\lambda \in \mathbb{R}$.

Further

Definition 4 $\mathbf{v_p} \bullet \mathbf{w_p} = (\mathbf{v} \bullet \mathbf{w})_p$ *and* $\mathbf{v_p} \times \mathbf{w_p} = (\mathbf{v} \times \mathbf{w})_{\mathbf{p}}$.

Definition 5 *If $\mathbf{v_p} \bullet \mathbf{w_p} = 0$ then $\mathbf{v_p}$ and $\mathbf{w_p}$ are **orthogonal.** The **norm** of $\mathbf{v_p}$ is*

$$\|\mathbf{v_p}\| = (\mathbf{v_p} \bullet \mathbf{v_p})^{1/2} = (\mathbf{v} \bullet \mathbf{v})^{1/2} = \|\mathbf{v}\|.$$

So $\mathbf{v_p}$ is a **unit** vector if $\|\mathbf{v_p}\| = 1$.

Definition 6 *If $\mathbf{e_p^1}, \mathbf{e_p^2}, ..., \mathbf{e_p^n} \in T_{\mathbf{p}}(\mathbb{R}^n)$ are n mutually orthogonal unit vectors (so $\mathbf{e_p^i} \bullet \mathbf{e_p^j} = \delta_{ij}$ for all $1 \le i, j \le n$), then they form a **frame** in $T_{\mathbf{p}}(\mathbb{R}^n)$.*

*The **natural frame** is $U_{i\mathbf{p}} = (0, ..., 0, 1, 0, ..., 0)^T_{\mathbf{p}}$, with 1 in the i-th position, 0 elsewhere, $1 \le i \le n$.*

If $\mathbf{p} = \mathbf{0}$ we drop $\mathbf{p}$ from the notation.

Recall that if $\{\mathbf{e}^i\}_{1 \le i \le n}$ is a frame of $\mathbb{R}^n$ then, given a vector $\mathbf{v}$, we have

$$\mathbf{v} = \sum_{i=1}^{n} \left(\mathbf{v} \bullet \mathbf{e}^i\right) \mathbf{e}^i.$$

Then

$$\begin{aligned}
\mathbf{v_p} &= \left(\sum_{i=1}^{n}\left(\mathbf{v}\bullet\mathbf{e}^i\right)\mathbf{e}^i\right)_p = \sum_{i=1}^{n}\left(\left(\mathbf{v}\bullet\mathbf{e}^i\right)\mathbf{e}^i\right)_{\mathbf{p}} \qquad \text{by definition 3,} \\
&= \sum_{i=1}^{n}\left(\mathbf{v}\bullet\mathbf{e}^i\right)\mathbf{e}^i_{\mathbf{p}} \qquad \text{by definition 3,} \\
&= \sum_{i=1}^{n}\left(\mathbf{v_p}\bullet\mathbf{e}^i_{\mathbf{p}}\right)\mathbf{e}^i_{\mathbf{p}} \qquad \text{by definition 4.}
\end{aligned}$$

Most of this course is restricted to $\mathbb{R}^3$. For example

Lemma 7 *Assume* $\mathbf{v_p}, \mathbf{w_p} \in \mathbb{R}^3$*. If* $\|\mathbf{v_p}\| = \|\mathbf{w_p}\| = 1$ *and* $\mathbf{v_p}\bullet\mathbf{w_p} = 0$ *then* $\{\mathbf{v_p}, \mathbf{w_p}, \mathbf{v_p}\times\mathbf{w_p}\}$ *is a frame at* $\mathbf{p}$.

Proof For vectors at the origin,

$$\begin{aligned}
\|\mathbf{v}\times\mathbf{w}\|^2 &= (\mathbf{v}\times\mathbf{w})\bullet(\mathbf{v}\times\mathbf{w}) \\
&= \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}\bullet\begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} \\
&= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \qquad (1) \\
&= v_2^2w_3^2 - 2v_2w_3v_3w_2 + v_3^2w_2^2 + v_3^2w_1^2 - 2v_3w_1v_1w_3 + v_1^2w_3^2 \\
&\qquad + v_1^2w_2^2 - 2v_1w_2v_2w_1 + v_2^2w_1^2 \\
&= \left(v_1^2 + v_2^2 + v_3^2\right)\left(w_1^2 + w_2^2 + w_3^2\right) - (v_1w_1 + v_2w_2 + v_3w_3)^2 \qquad (2) \\
&= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\bullet\mathbf{w})^2 .
\end{aligned}$$

Therefore, at a general point $\mathbf{p}$,

$$\begin{aligned}
||\mathbf{v_p}\times\mathbf{w_p}||^2 &= \left\|(\mathbf{v}\times\mathbf{w})_{\mathbf{p}}\right\|^2 = \|\mathbf{v}\times\mathbf{w}\|^2 \\
&= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\bullet\mathbf{w})^2 \\
&= \|\mathbf{v_p}\|^2\|\mathbf{w_p}\|^2 - (\mathbf{v_p}\bullet\mathbf{w_p})^2 .
\end{aligned}$$

Then $\|\mathbf{v_p}\| = \|\mathbf{w_p}\| = 1$ and $\mathbf{v_p} \bullet \mathbf{w_p} = 0$ implies $||\mathbf{v_p} \times \mathbf{w_p}|| = 1$.

For the orthogonality it is well known that

$$\mathbf{v} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = 0$$

Similarly $\mathbf{w} \bullet (\mathbf{v} \times \mathbf{w}) = 0$. The same results hold for vectors with a point of application. ■

Unnecessary aside. It is not necessarily obvious that (1) will rearrange as (2). Perhaps another approach is more obvious. First note that the cross product can be written as matrix multiplication, so

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = M_{\mathbf{w}} \mathbf{v},$$

say. Then

$$\|\mathbf{v} \times \mathbf{w}\|^2 = (M_{\mathbf{w}} \mathbf{v})^T (M_{\mathbf{w}} \mathbf{v}) = \mathbf{v}^T M_{\mathbf{w}}^T M_{\mathbf{w}} \mathbf{v}.$$

Here

$$\begin{aligned} M_{\mathbf{w}}^T M_{\mathbf{w}} &= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} w_3^2 + w_2^2 & -w_2 w_1 & -w_3 w_1 \\ -w_2 w_1 & w_3^2 + w_1^2 & -w_3 w_2 \\ -w_3 w_1 & -w_3 w_2 & w_2^2 + w_1^2 \end{pmatrix} \\ &= \left(w_1^2 + w_2^2 + w_3^2\right) I_3 - \begin{pmatrix} w_1^2 & w_2 w_1 & w_3 w_1 \\ w_2 w_1 & w_2^2 & w_3 w_2 \\ w_3 w_1 & w_3 w_2 & w_3^2 \end{pmatrix} \\ &= \|\mathbf{w}\|^2 I_3 - \mathbf{w}\mathbf{w}^T. \end{aligned}$$

Then

$$\begin{aligned}
\|\mathbf{v}\times\mathbf{w}\|^2 &= \mathbf{v}^T M_{\mathbf{w}}^T M_{\mathbf{w}} \mathbf{v} \\
&= \|\mathbf{w}\|^2 \mathbf{v}^T\mathbf{v} - \mathbf{v}^T\mathbf{w}\mathbf{w}^T\mathbf{v} \\
&= \|\mathbf{w}\|^2 \|\mathbf{v}\|^2 - (\mathbf{v}\bullet\mathbf{w})^2 .
\end{aligned}$$

Definition 8 *A* ***Vector Field*** V *on* $\mathbb{R}^n$ *is a function that assigns to each point* $\mathbf{p}$ *of* $\mathbb{R}^n$ *a tangent vector* $V(\mathbf{p})$*, so* $V(\mathbf{p}) \in T_{\mathbf{p}}(\mathbb{R}^n)$.

Given two vector fields V and W we can

Definition 9 *Define* $V+W$ *by* $(V+W)(\mathbf{p}) = V(\mathbf{p})+W(\mathbf{p})$ *for all* $\mathbf{p}\in\mathbb{R}^n$, *If* $f:\mathbb{R}^n\to\mathbb{R}$ *then* fV *is defined by* $(fV)(\mathbf{p}) = f(\mathbf{p})\,V(\mathbf{p})$ *for all* $\mathbf{p}\in\mathbb{R}^n$.

(So the set of vector spaces forms a vector space over the set of scalar-valued functions.)

Further

Definition 10 *Define* $V\bullet W$ *by* $(V\bullet W)(\mathbf{p}) = V(\mathbf{p})\bullet W(\mathbf{p})$ *for all* $\mathbf{p}\in\mathbb{R}^n$, *and define* $V\times W$ *by* $(V\times W)(\mathbf{p}) = V(\mathbf{p})\times W(\mathbf{p})$ *for all* $\mathbf{p}\in\mathbb{R}^n$.

Example 11 *Important vector fields on* $\mathbb{R}^n$ *are* U_i, $1\le i\le n$, *given by* $U_i(\mathbf{p}) = U_{i\mathbf{p}}$ *for all* $\mathbf{p}\in\mathbb{R}^n$ *and* $1\le i\le n$. *That is,* U_i *gives the* i*-th usual basis vector at* $\mathbf{p}$.

Definition 12 *The set* $\{U_i(\mathbf{p})\}_{1\le i\le n}$ *is the* ***natural frame*** *of* $T_{\mathbf{p}}(\mathbb{R}^n)$.

Lemma 13 *If* V *is a vector field on* $\mathbb{R}^n$ *then there are* n *uniquely defined functions* $v_i:\mathbb{R}^n\to\mathbb{R}$ *such that* $V=\sum_{i=1}^n v_i\,U_i$.

Proof For each $\mathbf{p}\in\mathbb{R}^n$, $V(\mathbf{p})\in T_{\mathbf{p}}(\mathbb{R}^n)$ so there exist real numbers $v_i(\mathbf{p})$ for $1\le i\le n$ such that

$$V(\mathbf{p}) = \sum_{i=1}^n v_i(\mathbf{p})\,U_{i\mathbf{p}} = \sum_{i=1}^n v_i(\mathbf{p})\,U_i(\mathbf{p}) = \left(\sum_{i=1}^n v_i\,U_i\right)(\mathbf{p})\,.$$

Doing this for each $\mathbf{p}\in\mathbb{R}^n$ defines the functions $v_i:\mathbb{R}^n\to\mathbb{R}$ for which we have

$$V = \sum_{i=1}^n v_i\,U_i.$$

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